Gnomons Land

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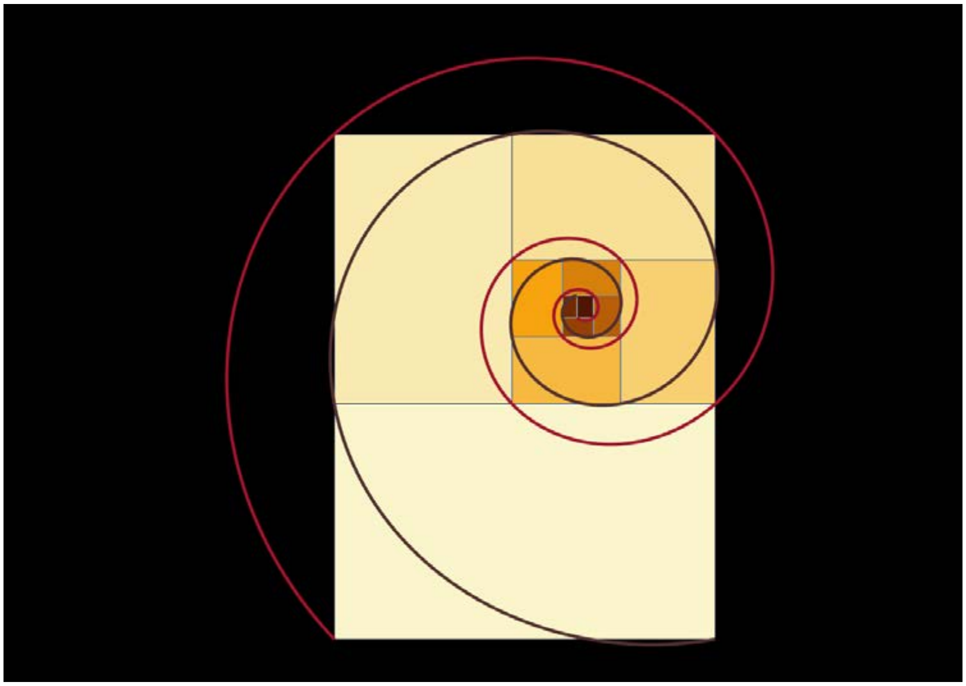

Figure 1: A dignomonic tiling and associated spirals

Abstract

Gazalé [1] has developed a system of whorled dignomonic tilings and corresponding spirals that intersect half of the tile vertices. We have developed a second spiral that complements the first in the sense that it intersects the remaining vertices. We determine the specific conditions when the complement is normal and tangent to the bounding tiles, that is, the complement is totally contained within the tiles. In addition, we demonstrate that when the dignomonic tiles devolve into a monognomonic tiling that the two spirals are the same logarithmic spiral.

Background

Whorled figures are well known, albeit perhaps not by that name. The Fibonacci tiling shown in Figure 2 is an example. The figure is built up by adding squares in a spiral manner such that there are no unfilled spaces. In this case, the tiles grow in size as the Fibonacci sequence. In general, the first tile is called the *seed* and the remaining tiles are all of the same shape and grow in size in such a manner as to leave no unoccupied territory; they are tessellated. Other options include permitting two or more shapes that accomplish the same goal.

A special case occurs when each added tile (or tiles) produces a composite image that is self-similar to the seed. In that case, the tile (or tiles) is called a *gnomon.* The term gnomon is generally taken to mean the vertical part of a sundial that casts the shadow, but in the present work we use it in the original ancient Greek context as a figure that, when added to another figure, produces the figure similar to the original [1]. An example of a (mono) gnomonic tiling is seen in Figure 6. Here, the seed is the golden rectangle; the gnomon is a square whose sides grow as the golden ratio, $\varphi = \left(1+\sqrt{5}\right)/2$. By contrast, Figure 1 shows a dignomonic tiling, that is, as two successive tiles are added the composite shape is self-similar to the seed.

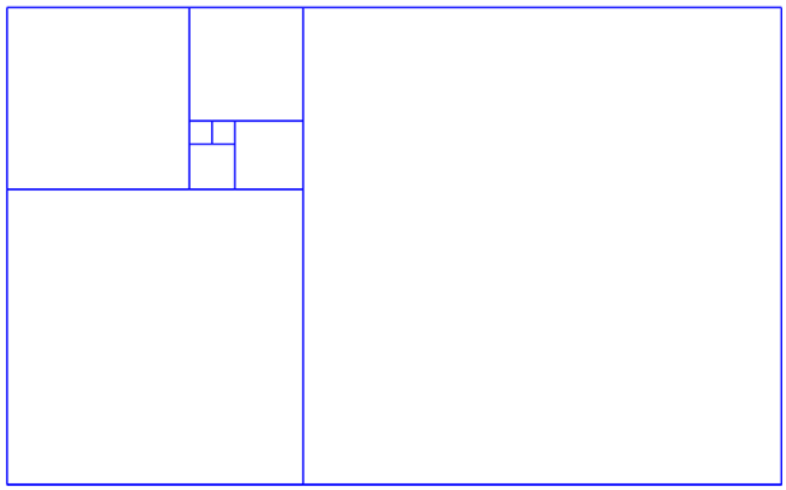

Figure 2: Fibonacci tiling.

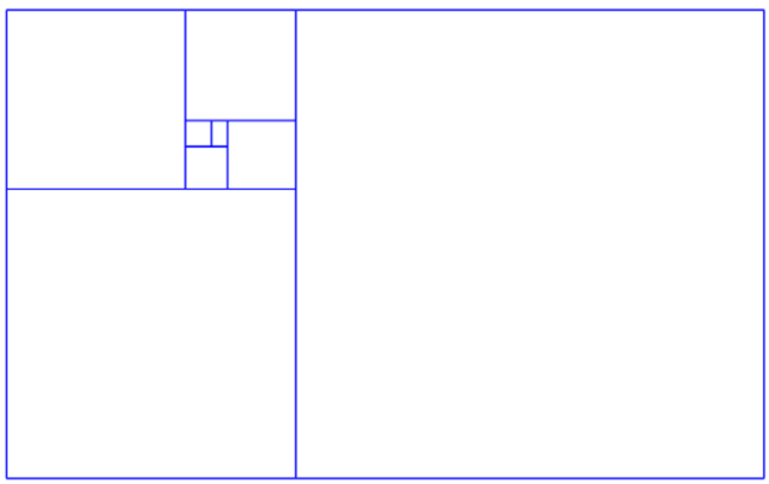

Figure 3: Golden ratio tiling.

Gazalé has formalized determination of the gnomons for mono- and dignomonic tilings in terms of an arbitrary seed. He then develops a spiral that begins at the seed and touches all the gnomons at two of the vertices. This is seen clearly in Figure 1, where the red spiral is Gazalé’s. This will be described in the next section.

Technical

Since the monognomonic tiling, exemplified in Figure 3 is a subset of the dignomonic tiling we shall just jump into the dignomonic tiling straightaway. Figure 4 shows the setup the dignomonic tiling.

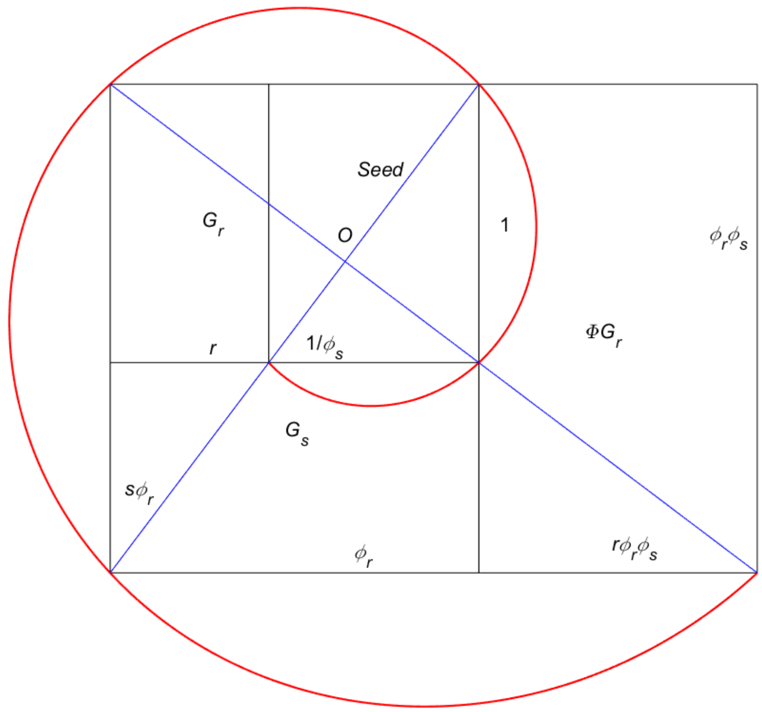

Figure 4: Gazalé's dignomonic tiling and spiral.

The entire tiling is dependent on only two parameters, φ_s, which defines the seed, and φ_r. The two gnomons, denoted by G_r and G_s, are dependent only on these parameters. The distances *r* and *s* in Figure 4 are given by

$$r = \varphi_r - 1/\varphi_s ; \quad s = \varphi_s - 1/\varphi_r \tag{1}$$

In addition, as the pair of tiles is added sequentially in an anticlockwise direction, they increase in size by $\Phi = \sqrt{\varphi_r \varphi_s}$. The skewed blue axes that pass through two vertices of each rectangle determine the path of the spiral. It is worth noting that these lines are not generally normal to each other. Gazalé [1] develops the equations for the spiral; the result is

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{b\theta} \begin{bmatrix} \cos\theta & -\sqrt{\tfrac{r}{s}}\sin\theta \\ \sqrt{\tfrac{s}{r}}\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$x_0 = -\frac{1}{\varphi_s\left(1+\varphi_r\varphi_s\right)}; \quad y_0 = -\frac{1}{1+\varphi_r\varphi_s} \tag{2}$$

where $b = 2\ln\Phi/\pi$ is the logarithmic constant, or flair coefficient. There are two things that are noteworthy here. First, Eq. (2) applies to a Cartesian coordinate system with origin at O. (The coordinate system for the tiling is centered at (x_0, y_0), that is, the starting point of the spiral.) Second, while the spiral definitely has a logarithmic growth rate, it is decidedly not a logarithmic spiral because of the skew terms in the rotation matrix of Eq. (2).

Now, we look at Figure 4 and wonder, why not the spiral in Figure 5? It seems to be perfectly viable and if we follow Gazalé's derivation we wind up with exactly the same as Eqs. (2) with the sole exception that

$$y_0 = 1 - \frac{1}{1+\varphi_r\varphi_s} \tag{3}$$

We call this a complementary spiral in the sense that it complements Gazalé's spiral by touching all the remaining vertices. Notice that in general this spiral is skew to the tile borders.

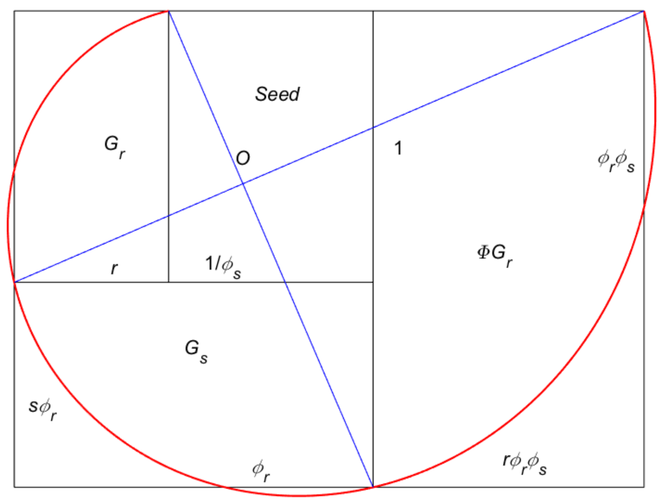

Figure 5: Gazalé's dignomonic tiling and spiral.

So this begs the question, is the complementary spiral ever confined to the tiles for a dignomonic tiling (and, by extension for a monognomonic tiling as well)? That is, is it ever normal and tangent to the tile borders? It turns out to be sufficient that the starting point of the complementary spiral is tangent to the bounding rectangle. Thus, from Eqs. (2) and (3), we find

$$\left.\frac{dy}{dx}\right|_{\theta=0} = \frac{\sqrt{\frac{s}{r}}x_0 + by_0}{bx_0 - \sqrt{\frac{r}{s}}y_0} = 0 \tag{4}$$

Thus, setting the numerator equal to zero, and substituting for x_0, y_0, and b and taking note of the additional relation

$$\sqrt{\tfrac{s}{r}} = \sqrt{\tfrac{\varphi_s - 1/\varphi_r}{\varphi_r - 1/\varphi_s}} = \sqrt{\tfrac{\varphi_r\varphi_s - 1}{\varphi_r\varphi_s - 1}\tfrac{\varphi_s}{\varphi_r}} = \sqrt{\tfrac{\varphi_s}{\varphi_r}} \tag{5}$$

we find that

$$\sqrt{\tfrac{s}{r}}x_0 + by_0 = -\sqrt{\tfrac{\varphi_s}{\varphi_r}}\tfrac{1}{\varphi_s(1+\varphi_r\varphi_s)} + \tfrac{2}{\pi}\ln\left(\sqrt{\varphi_r\varphi_s}\right)\tfrac{\varphi_r\varphi_s}{(1+\varphi_r\varphi_s)} = 0 \tag{6}$$

Or, more simply

$$\tfrac{2}{\pi}\Phi_{crit}^3 \ln\Phi_{crit} - 1 = 0 \tag{7}$$

We find that $\Phi_{crit} = 1.538862046790905$. Thus, for any arbitrary seed defined by φ_s, in mono- or dignomonic tiling, we can choose $\varphi_r = \Phi_{crit}^2/\varphi_s$ to assure that the complementary spiral is confined to the tiling. Figure 6 shows such a spiral for an arbitrary value of φ_s.

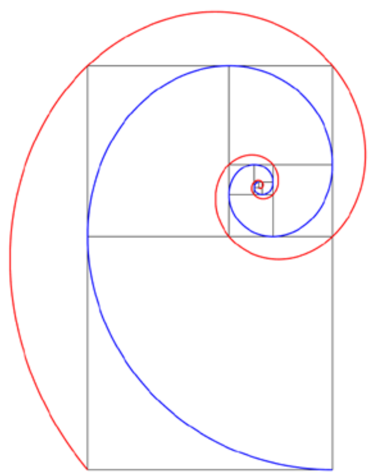

Figure 6: Example of a complementary spiral perfectly bound by the tiling.

Now, we mentioned previously that monognomonic tiling is a subset of dignomonic tiling. Now we can see how that comes to be. When the driving parameters φ_s and φ_r are the same, then $\Phi = \varphi_r = \varphi_s$, and G_s and G_r are self-similar, although the latter is larger by a factor of Φ. In other words there is a single gnomon. Here too, the complementary spiral is generally skew to the tile borders, except when $\varphi_s = \Phi_{crit}$. Moreover, the spiral axes are normal to each other and they are true logarithmic spirals. In fact, they are both part of the same logarithmic spiral; they differ only in angular range and rotation about the native origin ***O*** in the plane. This is difficult to do analytically because the two spirals have different angular ranges. However, we can demonstrate it graphically. First we must determine the angle to rotate the spiral. The setup is shown in Figure 7.

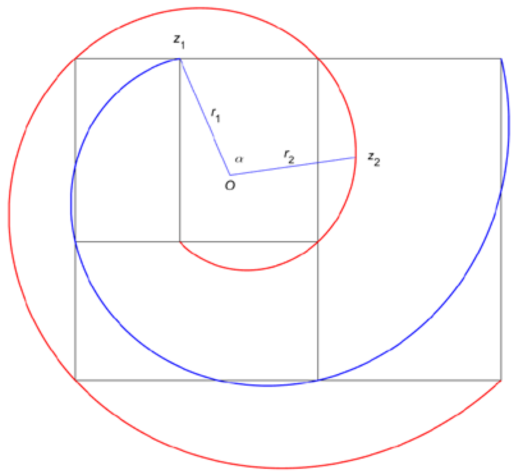

Figure 7: Geometry for spiral rotation.

Thinking in terms of the native spiral coordinates centered at ***O***, find the point on the red spiral where the distance $r_2 = r_1$. Then, from the scalar product expressed in complex variables

$$\alpha = \cos^{-1}\left(\frac{\mathrm{Re}\left(z_1 z_2^*\right)}{r_1 r_2}\right) \tag{8}$$

Then, if we take the outer (red) spiral and rotate it anticlockwise by α it will completely, and exactly cover the inner (blue) spiral. This is demonstrated in Figure 8, where that rotation is shown by the green spiral.

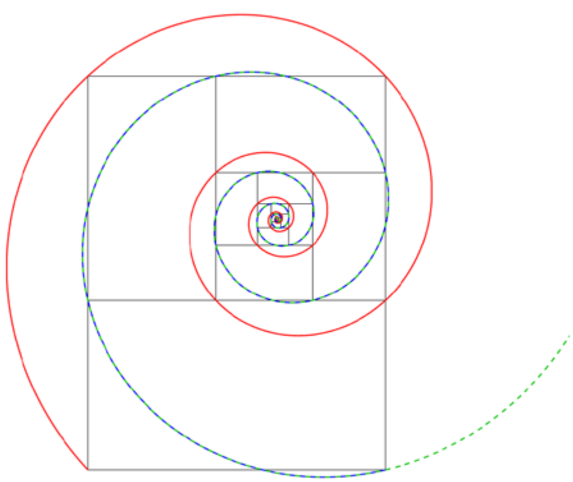

Figure 8: Demonstration of the self-similarity of the two spirals of a monognomonic tiling..

There is another way, albeit trivial, to convert a dignomonic to monognomonic tiling. And that is by simply combining the tiles G_s and G_r into a single *L*-shaped tile. In this case, each successive tile is increased in size by Φ^2 rather than Φ. Figure 9 shows a comparison of dignomonic tiling and its monognomonic counterpart. The spirals are the same for both tilings.

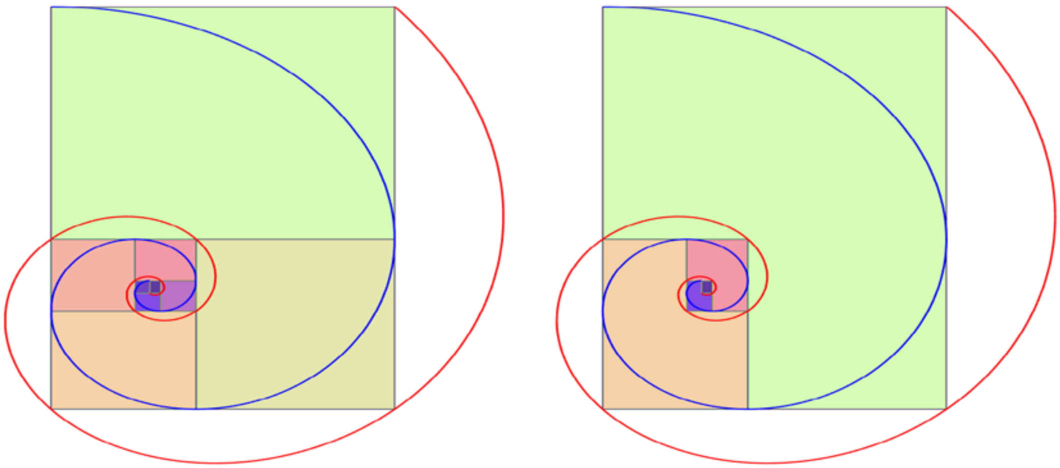

Figure 9: Comparison of a dignomonic tiling and it monognomonic counterpart.

Computer program

A Matlab computer program to produce the mono- and dignomonic tilings in this paper is provided in the Appendix. It's quite simple to use. If there are no input parameters beyond the number of tiles desired, the program produces a monognomonic tiling with $\varphi_s = \varphi_r = \Phi_{crit}$. If there is one input additional parameter, the program produces a normal monognomonic tiling with $\varphi_s = \varphi_r$. If there are two more input parameters (φ_s and φ_r, respectively) the program produces a regular dignomonic tiling; however, if the second input is 'empty,' that is, input as [], then the program will set $\varphi_r = \Phi_{crit}^2 / \varphi_s$. In all cases, Gazalé's spiral and its complement are drawn over the tiling. The only requirement is that the growth rate $\Phi = \varphi_r \varphi_s > 1$.

Summary and Conclusions

We have made a modest addition to Gazalé's work on dignomonic tiling and spiral. We have demonstrated an additional spiral that complements Gazalé's in that it crosses all the other vertices. Moreover, we analytically determined the conditions for the complement to be totally contained within the tiling and shown that the two spirals are self-similar in the monognomonic case. For additional interesting work on gnomons and tiling, see Waldman and Gray [2].

References

[1] M.J. Gazalé, *Gnomon: From Pharaohs to Fractals*. Princeton University Press (1999).
[2] Waldman, C.H. and Gray, S.B, Fibonacci, Padovan, & Other Pseudospirals, submitted for publication (2016).

Appendix: Computer Program

```
function DignomonicTiling4NCB(tiles,varargin)
% this program calculates mono- and dignomonic tilings and spirals
% USAGE
% DignomonicTiling4NCB(tiles): monognomonic with phis=Phi_crit
% DignomonicTiling4NCB(tiles,phis): regular monognomonic
% DignomonicTiling4NCB(tiles,phis,phir): regular dignomonic
% DignomonicTiling4NCB(tiles,phis,[]); dignomonic with phi2=Phi_crit^2/phis
% tiles = number of tiles in the tessellation
% Copyright 2016, CYe H. Waldman, algorithmicart@att.net
Phi_crit=1.538862046790905; % Phi for enclosed comp spirals; +/-1e16

if nargin==1
   phis=Phi_crit; phir=phis;
elseif nargin==2;
   phis=varargin{1};
   phir=phis;
elseif nargin==3
   phis=varargin{1};
   phir=varargin{2};
   if isempty(phir)
      phir=Phi_crit^2/phis;
   end
end

% test for suitability
if phis*phir<=1
   warning('Must have phis*phir > 1; they have both been inverted')
   phis=1/phis;    phir=1/phir;
end
% tiles
Z=DignomonicTiles(tiles,phis,phir);
% spirals
npts=100001;
[Zs,Ws]=DignomonicSpirals(tiles,phis,phir);
z0=Zs(1);
w0=Ws(1);
% showtime
```

```
figure;plot(Z,'k');axis equal
hold on;plot(Zs-z0,'r','LineWidth',1);
hold on;plot(Ws-w0+i,'b','LineWidth',1);
Phi=sqrt(phis*phir);
title(['[{\it\phi_s},{\it\phi_r},\Phi] = [' num2str(phis) ',' num2str(phir) ',' num2str(Phi) ']'])

return

function Z=DignomonicTiles(tiles,phis,phir)
% generate the tiling in cartesian coords
iseven=inline('rem(number,2) == 0;','number');
wrapN = @(x,N) (1+mod(x-1,N));

% three basic rectangle: seed, Gr, Gs
r=phir-1/phis; s=phis-1/phir;
seed=[0;1/phis;1/phis+i;i;0];
Gr=[0;r;r+i;i;0];
Gs=[0;phir;phir+i*s*phir;i*s*phir;0];

z1=seed;
z2=Gr-r;
z3=Gs-r-i*s*phir;
Z=[z1 z2 z3];
prevG=z3;
for k=4:tiles
   if iseven(k)
      G=Gr*(phir*phis)^((k-2)/2);
   else
      G=Gs*(phir*phis)^((k-3)/2);
   end
   n1=wrapN(k+1,4);
   n2=wrapN(k+2,4);
   z=G-G(n1)+prevG(n2);
   Z=[Z z];
   prevG=z;
end

return
```

```
function [Zs,Ws]=DignomonicSpirals(tiles,phis,phir)
% generate the spirals in the native coordinates
% create the spiral as per Gazale, p. 165 ff
% common elements
r=phir-1/phis;
s=phis-1/phir;
m=sqrt(r*s);
Phi=sqrt(phir*phis);
b=2*log(Phi)/pi;
npts=100001;

theta=linspace(0,(tiles+1)*pi/2,npts)';
cost=cos(theta);
sint=sin(theta);
y0=-1/(1+phir*phis);
x0=y0/phis;
% x,y,R in spiral coords
x=exp(b*theta).*(x0*cost-y0*r/m*sint);
y=exp(b*theta).*(x0*s/m*sint+y0*cost);
Zs=x+i*y;

% the complemntary spiral
theta=linspace(0,(tiles-1)*pi/2,npts)';
cost=cos(theta);
sint=sin(theta);
v0=-1/(1+phir*phis);
u0=v0/phis;
v0=1+v0;
% x,y,R in spiral coords
u=exp(b*theta).*(u0*cost-v0*r/m*sint);
v=exp(b*theta).*(u0*s/m*sint+v0*cost);
Ws=u+i*v;

return
```